Geographical Research Forum
No. 6, September, 1983.

OPTIMIZING THE SPACING OF FEEDER ROAD SYSTEMS

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Submitted April 1983, revised June 1983.

This paper examines the optimal spacing of a set of feeder roads emanating from a single penetration road which has previously been extended into the hinterland of a port. The specification of the optimal structure is based on an analysis of the relationship between the cost of feeder road construction and the increases in land rent they provide. The feeder road structure is completely specified by the distances between successive feeders along the main penetration road. These spacings can be determined by solving a set of non-linear simultaneous equations. Numerical analysis can then be used to evaluate the sensitivity of these optimal spacings (including the number of feeders to be built) to changes in the key parameters of on and off-road transport rates and the per mile construction cost of feeder roads.

We now know a fair amount about the optimal geometrical structure of transport networks, at least in some well-defined cases (see, for example, Beckmann, 1952; Hauer, 1972; Puu, 1978; Sen, 1971; Smith, 1976, 1979; Tanner, 1967, 1970; Werner, 1968; and Werner and Boukidis, n.d.). Taken as givens the distribution of trips over the network and the network topology, some characteristic of network structure is then determined by optimizing a specific objective function. Smith (1979), for example, assumes that a fixed origin-destination distribution of trips utilizes a given radially structured road network and then determines the location of the single ring road which maximizes traffic relief to the radials. In a significantly different approach, Ralston and Barber (1982) have shown how the optimal dynamic structure of a single road (in this case the length) can be determined by evaluating the benefits of road construction in terms of land rent impacts. It is proposed here to utilize this approach in the specification of the optimal structure of a feeder road system.

Despite the importance of feeder roads in development economics, this problem has received only passing attention, particularly at the theoretical level. Under what conditions will feeder roads emerge? What are the likely optimal structures of feeder road systems in specific idealized environments? What are the dynamics by which joint arterial-feeder road systems emerge over time? The purpose of this paper is to make a modest beginning to the provision of answers to such questions. It will begin with a brief description of a simple idealized environment for the feeder road problem, a neo-Von Thunen situation previously utilized by Walters (1968) in his analysis of road user charges. It is then shown how the optimal locations of a single feeder road and a two-feeder system can be determined. These results are subsequently generalized to any multiple feeder system and it is shown that it is possible to determine both the optimal number and location of feeder roads in a complete system. Numerical analysis is then used to examine the changes in optimal feeder structure to the key parameters used in the model. The paper closes with a summary of the principal results and speculation concerning dynamic aspects of the feeder road problem.

AN IDEALIZED FEEDER ROAD ENVIRONMENT

Consider a port P located on a straight coastline oriented north-south (see Figure 1). In the hinterland surrounding this port a single crop is grown and transported to the market at P where it is sold at a given fixed price of \$k per ton. The land is homogeneous in all respects and, without loss of generality, will be assumed to have a unit yield. A single arterial PZ has previously been extended from the port to the limit of cultivation in a direction perpendicular to the coast. Per mile transport costs along this road are \$a per ton-mile, less than the off-road rate of \$b per ton-mile. Transport is limited to the east-west and north-south directions[1]. Under these conditions, production in the hinterland is limited to the triangle QRZ where Z is k/a miles from the port P and Q and R are k/b miles from P.

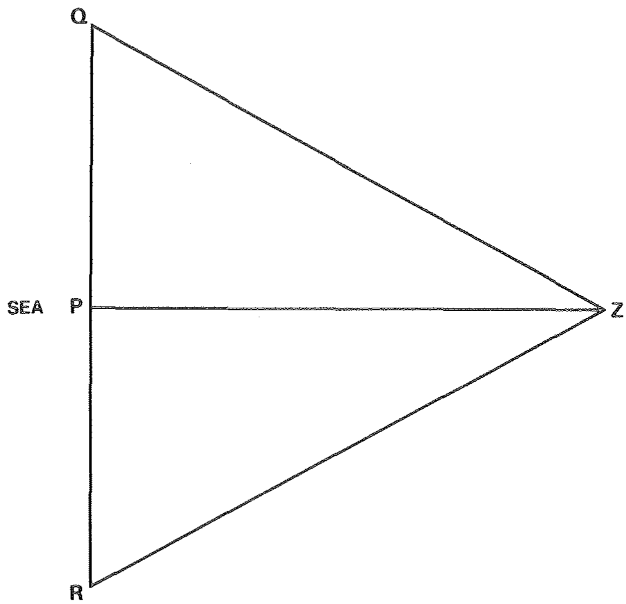

Fig. 1: The Scenario for the Walters' Model

Implicit in this situation is a system of land rents created by differential access to the port P. Rent at any point is given as the difference between the price of the commodity at the port and the total transport charge involved in bringing the commodity to the market at P. In the initial situation, rents will appear as a half-pyramid with height k at P and diminishing linearly to zero at points Q, R, and Z. The total initial rent is equal to the volume of this half-pyramid or $k^3/3ab$. Let us assume that any feeder roads extended into the hinterland off the main arterial PZ have their *total* benefits capitalized in terms of increments to this initial land rent condition. An *optimal* feeder road system maximizes total net benefits, the difference between total incremental land rent and the costs of feeder road construction and maintenance.

The selection of the optimal number and locations of feeder roads off the main arterial PZ involves several trade-offs. First, the greater the number of feeder roads and the greater the total length of the feeders the greater the annual costs of construction and maintenance. However, fewer, shorter feeder roads invariably will result in smaller increases to the area under cultivation and less incremental land rent. However, this simple trade-off is complicated by the fact that the impacts of the individual feeder roads are interdependent and there is a considerable "wastage" of feeder road mileage. First, each feeder must pass through the cultivated area PRZ which is already serviced by the main arterial. Secondly, when the feeders are spaced too closely their individual hinterlands also overlap. These issues will be fully explored in the subsequent analysis which proceeds logically from the simplest single feeder road case, to a two feeder system, and ultimately to most general case of N feeder roads.

A SINGLE FEEDER ROAD

Suppose it has been decided to augment the existing arterial PZ with a single feeder road built perpendicular to the arterial at a distance of d miles from the port (see Figure 2)[2]. Further suppose that the feeder AC is built to the same standard as the arterial and is extended to the new limit of cultivation[3]. The length of AC is thus $k/a - d$ miles. The closer it is built to the port the greater the length of the feeder[4].

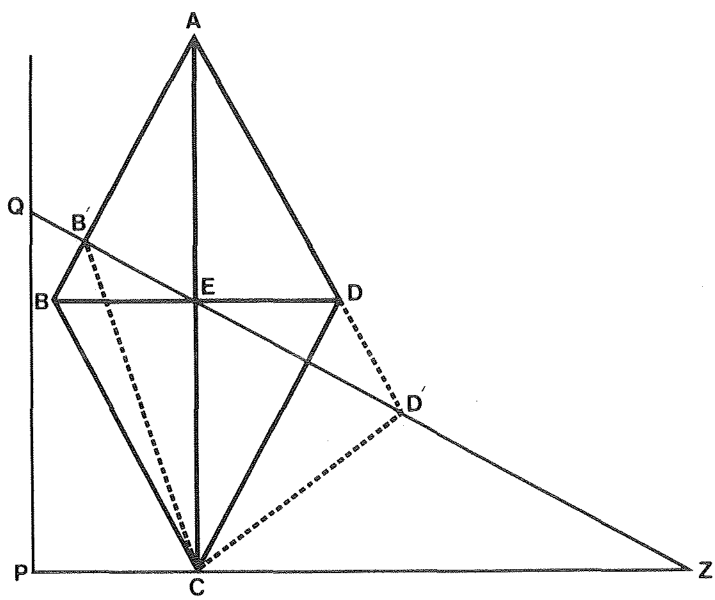

Fig. 2: The Impact of a Feeder Road on the Cultivated Area

What will be the effect of this new feeder? First, the cultivated area will be extended to include the new triangular area AB'D'. Land rent in this area will increase. Secondly, the land rent in the area B'CD' also increases because of the improved accessibility to the port offered by the new feeder. The increases in land rent are most clearly illustrated in the three dimensional rent pyramids of Figure 3.

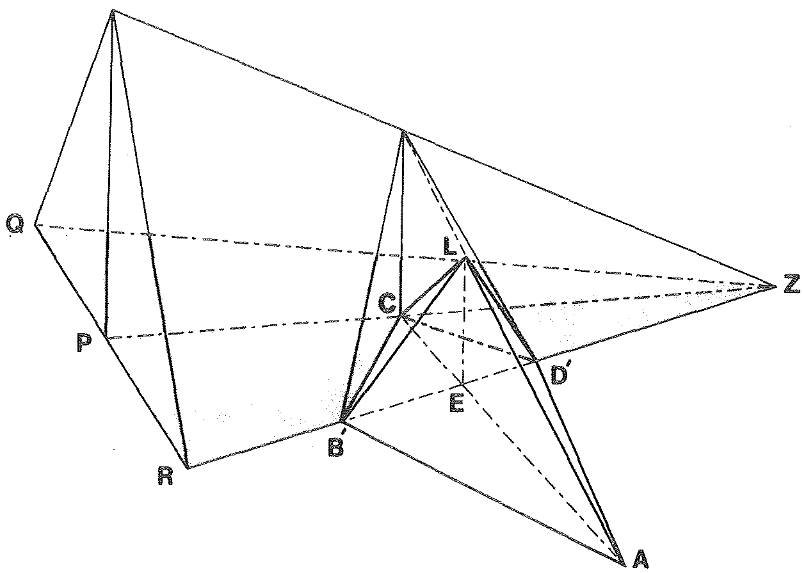

Fig. 3: The Impact of a Feeder Road on Land Rent

The volume of the half-pyramid based on triangle PRZ represents the initial land rent before any feeder roads have been constructed. The *incremental* rent due to the construction of the feeder road AC is given as the sum of the "half-pyramids" AB'D' and B'CD' each of which has a maximum at the intersection of the feeder road and the initial limit of cultivation, point E. As the location of the feeder road AC is moved along PZ, the length of the feeder (and hence its cost) and the volume of the created benefits varies. The problem is to determine the distance from the port where set benefits are maximized.

For purposes of simplification, however, it is assumed that the initial limit of cultivation QZ is perpendicular to the feeder road in the vicinity of point E. It is then possible to approximate the impacted area as the diamond-shaped ABCD rather than AB'D'. This turns out to be quite a reasonable assumption and is increasingly accurate as the ratio of b:a increases. The boundaries AB and AD are determined by the condition that the total transport cost to the port using the feeder just equals $k. The boundaries BC and CD are determined by the equation of the total transport cost to the port via the direct route involving only the main arterial and the route employing the feeder and the arterial.

In order to determine the optimal location of the feeder road, it is necessary to develop analytical expressions for the volume of the rent pyramids based on the diamond-shaped ABCD. First, let us impose a coordinate system with the port P as the origin and the major arterial as the x-axis. In terms of the parameters k, a, and b and the variable d we can show that BD=2(k-ad)(1-a/b)(1/b), EC=(k-ad)(1/b), AE=(k-ad)(1-a/b)(1/a) and the height of the pyramid at E is (k-ad)(1-a/b). Therefore, the total incremental land rent of a feeder road located d miles from the port can be expressed as

$$R(d) = 1/3 \frac{(b-a)^2}{b^3 a} (k-ad)^3 \qquad (1)$$

Of course it is possible this entire benefit may not be realized if the feeder road is built too close to the port as in Figure 4. If $d \leq d' = k(b-a)/(b^2+ab-a^2)$ some of this land rent will be "lost in the sea". This "lost rent", Q(d), is equal to the sum of the volumes of the quarter-pyramids based on BGH and BFG and therefore

$$Q(d) = \frac{1}{6\,a\,(b-a)} [(k-ad)(1-a/b) - db]^3 \qquad (2)$$

Net incremental rent is R(d)-Q(d).

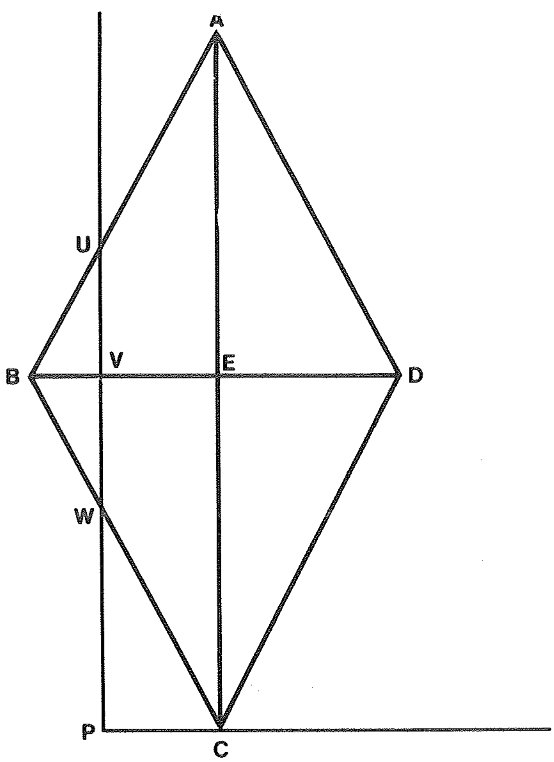

Fig. 4: The Possible Compliction if Feeder Road is Constructed Too

Close to the Port

The simplest of all functional forms is used for road construction and maintenance costs. The annual cost of feeder road construction and maintenance is assumed to be a constant \$c per mile. The total cost of a feeder road d miles from port P is

$$C(d) = c\ (k/a - d) \tag{3}$$

which is a linear, strictly decreasing function of distance from the port.

The single feeder road problem can now be formally expressed as

$$\underset{d \in [0,k/a]}{\text{MAX}} \quad R(d) - Q(d) - C(d) \tag{4}$$

subject to

$$R(d) - Q(d) - C(d) \geq 0 \tag{5}$$

Constraint (5) requires that the feeder yield positive net benefits. The situation is portrayed graphically in Figure 5. R(d) is a third degree polynomial over its entire range but strictly decreasing over [O,k/a]; it is illustrated as ABZ. Q(d) is given Dd'. OD is equal to one-half of OA since a feeder built on the coast will lose one-half of its total benefits (those on the left-hand side) to the sea. As the feeder road is built further and further inland, Q(d) declines at a continuously increasing rate. At a distance $d' = k(b-a)/(b^2+ab-a^2)$ the entire hinterland of the feeder road is inland. R(d)-Q(d) is given by DBZ. At B, R(d)-Q(d) meets R(d) smoothly and is identical with R(d) over the range [d', k/a].

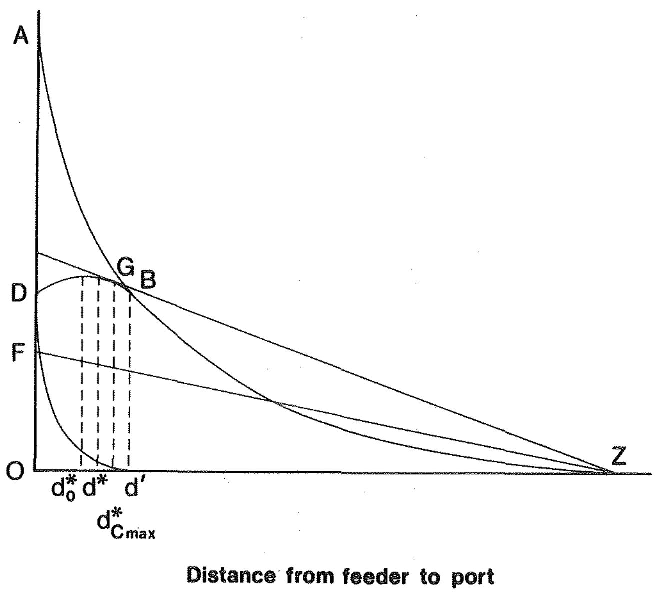

Fig. 5: Determining Optimal Location of a Single Feeder Rood

Let us suppose $c = 0$. In this case, C(d) would be given by OZ. The optimal value of d is d^*_0. Note that even when $c = 0$ the feeder is located so that *some* of its potential hinterland is "lost to the sea". Though rents decline with distance from P, the size of the actual hinterland of the feeder increases until $d = d'$. As c increases the slope of C(d) increases and can be illustrated, for example, as FZ. Here the optimal d is coincident with the maximum vertical difference between R(d)-Q(d) (DBZ) and C(d) or FZ. In this case the optimal value is labeled d^*. As c increases, the slope of FZ increases and at some point is just tangent to DBZ, at G. This is the maximum value of c at which a feeder road can pay for itself through rent increments, leading to a feeder road $d^*_{c_{max}}$ miles from the port at P. At values of c in excess of c_{max}, C(d) lies above R(d)-Q(d) and no feeder road can be built. Revenue from incremental rent is simply insufficient to offset the costs of feeder road construction and maintenance.

A clear rule describing the situations in which feeder roads can emerge can be easily developed from the formal conditions for an optimum to (4) and (5). Moreover, the equation of the marginal costs of road construction with the marginal incremental rent yields the optimal locations for feeder roads. Feeder roads can be built whenever $c < c_{max}$. Unfortunately, it is not easy to develop a simple analytical expression for d^* even though $R'(d) - Q'(d) - C'(d)$ is a quadratic function.

A TWO FEEDER SYSTEM

Consider now the two feeder system of Figure 6. Two feeders AC and FH have been extended into the hinterland of the main arterial to their respective limits of cultivation. AC is d_1 miles from the port, FH is d_2 miles from the port and CH is thus

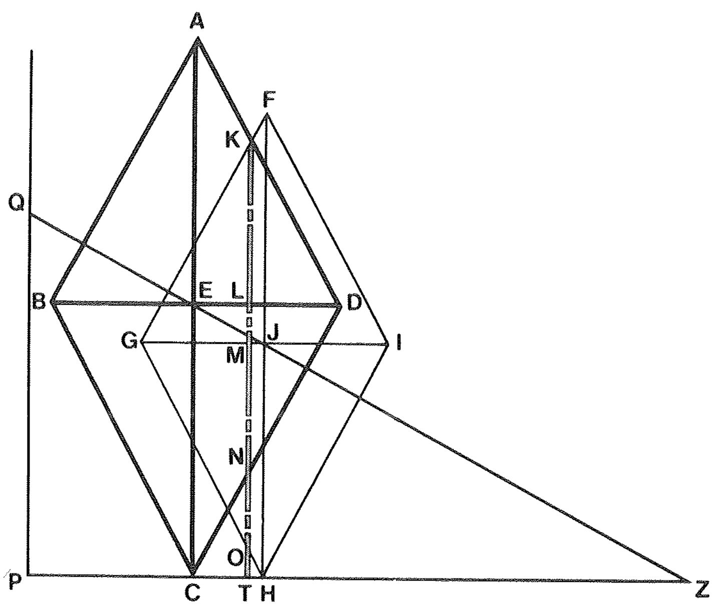

Fig. 6: The Impact of a Two-Feeder Road System on the Cultivated Area

d_2-d_1 miles in length. Approximating the hinterlands of the feeder roads by diamond-shaped areas ABCD and FGHI again simplifies the problem. The net incremental rent of the two feeder system can be divided into four components:

(1) $R(d_1,d_2)$--	the total incremental rent of the two hinterlands ABCD and FGHI;
(2) $Q(d_1)$--	the rent foregone if feeder AC is built too close to the port and its hinterland overlaps with the sea;
(3) $S(d_1 d_2)$--	the rent foregone if the two feeders are built too close together and have overlapping hinterlands;and
(4) $C(d_1,d_2)$--	the costs of feeder road construction and maintenance.

Let us consider each in turn.

$R(d_1,d_2)$: This is the sum of the incremental rent based on the two areas ABCD and FGHI. For ABCD, this is identical to equation (1) above and for FGHI we have

$$R(d_2) = \frac{1}{3}\,\frac{(b-a)^2}{b^3 a}\,(k - ad_2)^3 \qquad (6)$$

so that

$$R(d_1,d_2) = \frac{1}{3}\,\frac{(b-a)^2}{b^3 a}\,[(k - ad_1)^3 + (k - ad_2)^3] \qquad (7)$$

As will be shown below this component easily generalizes for the N feeder road system.

$Q(d_1)$: This is identical to (2). There are no terms for $Q(d_2)$ since any rent foregone if feeder 2 is built to close to the port will be subsumed under the overlap of the two feeders' hinterlands given below as $S(d_1,d_2)$.

$S(d_1,d_2)$: As is illustrated in Figure 6, the two feeders may have overlapping hinterlands. In fact, any optimal system for feeder roads must have overlapping hinterlands since their locations will always be drawn to the port where rent is at a maximum. First, we must determine the breakpoint between successive feeders. Farmers will ship their produce to the port utilizing the cheapest route. Excluding those farmers who will only use the main arterial and no feeders, we can define the breakpoint between successive feeders by the equation of total costs to P using either feeder road. It can be shown that the breakpoint between successive feeders, KT in Figure 6, is always a constant proportion of the inter-feeder spacing. This breakpoint occurs at $(a+b)/2b$ of the distance between any two feeders. Thus, all of the incremental rent based on the areas KND and KGO must be subtracted from $R(d_1,d_2)$.

The calculation of this overlapping rent involves the summation of the rent from four separate quarter-pyramids based on the triangles KMG, OMG, LND, KLD. It can be shown that

$S(d_1,d_2) =$

$$\frac{1}{6ab}\left[\left((k-ad_1)(1-a/b)-\frac{a+b}{2}(d_2-d_1)\right)^3+\left((k-ad_2)(1-a/b)-\frac{b-a}{2}(d_2-d_1)\right)^3\right]$$
$$+\frac{1}{6b}\left[\left((k-ad_1((1-a/b)-\frac{a+b}{2}(d_2-d_1)\right)^2\left((k-ad_1)(1/b)-\frac{a+b}{2(b-a)}(d_2-d_1)\right)\right.$$
$$\left.+\left((k-ad_2)(1-a/b)-\frac{b-a}{2}(d_2-d_1)\right)^2\left((k-ad_2)(1/b)-a/2(d_2-d_1)\right)\right] \qquad (8)$$

When $(d_2-d_1) = 0$, that is when the two feeders are located at the same distance from the port, it can be shown that $S(d_1,d_2) = R(d_1) = R(d_2)$ and there is a complete overlapping of hinterlands.

$C(d_2,d_1)$: Finally we must account for our expenditures on road construction and maintenance . Each feeder will cost $c(k/a - d_i)$, so that two feeders will cost

$$C(d_1,d_1) = c\ (2\ k/a - d_1 - d_2) \qquad (9)$$

Combining these components, the maximation of total net benefits can be expressed as

$$\underset{d_1,d_2}{\mathrm{MAX}}\quad F(d_1,d_2) = R\ (d_1,d_2) - Q(d_1) - S(d_1,d_2) - C(d_1,d_2) \qquad (10)$$

subject to

$$F\ (d_1,d_2) \geq 0 \qquad (11)$$

and

$$d_2 \geq d_1 \qquad (12)$$

Constraint (11) requires positive net benefits for feeder construction to take place and constraint (12) requires the second feeder road to be located no closer to the port than the first feeder road.

The maximization of (10) yields two simultaneous equations in d_1 and d_2 which can be solved using most available routines. Constraint (12) is met by starting the solution iterations with the constraint holding. Normally, the constraint will hold for any optimal solution located by an algorithm such as a Newton or modified Newton procedure.

MULTIPLE FEEDER ROAD SYSTEMS

This model can be easily generalized to simultaneously determine the optimal locations of N feeder roads spaced along the length of the main penetration road. Using the notation developed in previous sections, the specification of the optimal spacing of a set of N feeder roads $(d^*_1,d^*_2,\ldots,d^*_N)$ can be expressed as

$$\max_{d_i} F(d_1,d_2,\ldots,d_N) = R(d_1,d_2,\ldots d_N) - Q(d_1) - \sum_{i=1}^{N-1} S(d_i,d_{i+1})$$

$$- C(d_1,d_2,\ldots,d_N) \qquad (13)$$

where

$$R(d_1,d_2,\ldots,d_N) = \frac{(b-a)^2}{3b^3a}\left[\sum_{i=1}^{N} (k-ad_i)^3\right] \qquad (14)$$

$Q(d_1)$ is given by (2),

$$C(d_1,d_2,\ldots d_N) = c\left[N\frac{k}{a} - \sum_{i=1}^{N} d_i\right] \qquad (15)$$

and

$S(d_i,d_{i+1}) =$

$$\frac{1}{6ab}\left(((k-ad_i)(1-a/b)-\frac{a+b}{2}(d_{i+1}-d_i))^3+((k-ad_{i+1})(1-a/b)-\frac{b-a}{2}(d_{i+1}-d_i))^3\right.$$

$$+\frac{1}{6b}((k-ad_i)(1-a/b)-\frac{a+b}{2}(d_{i+1}-d_i))^2((k-ad_i)(1/b)-\frac{a+b}{2(b-a)}(d_{i+1}-d_i))$$

$$+\frac{1}{6b}((k-ad_{i+1})(1-a/b)-\frac{b-a}{2}(d_{i+1}-d_i))^2((k-ad_{i+1})(1/b)-a/2(d_{i+1}-d_i)) \qquad (16)$$

The summation of $S(d_i,d_{i+1})$ extends over N-1 overlaps of the N feeder roads. In addition, there is the implicit constraint set

$$d_{i+1} \geq d_i \qquad i = 1,2,\ldots,N-1 \qquad (17)$$

which requires that successive feeders be located at increasing distances from the port.

The solution to this problem yields N simultaneous equations

$$\frac{\partial F}{\partial d_i} = 0 \qquad i = 1,2,\ldots,N \qquad (18)$$

which is solved by any available routine. The proper choice of starting points for d_i will generally result in a set of optimal spacings $(d^*_1,d^*_2,\ldots,d^*_N)$ which satisfy (17), since the function is well behaved over the appropriate range d_i.

While the maximization of (13) yields the optimal spacings for a set of N feeder roads, it remains to determine the optimal *number* of feeder roads for any given set of parameters a, b, c, and k. This is achieved by solving (13) for progressively larger and larger systems of feeder roads and selecting the optimal overall system by inspection. Of course when c = 0, $F(d_i)$ is maximized when there is an infinite number of feeder roads with zero spacing. However, for any positive value of c there will always be a finite number of feeder roads in the optimal system. As the number of feeders increases, the increase in construction and maintenance costs as well as the overlap of feeder road hinterlands leads to a point where net benefits actually decrease with increasing N.

A NUMERICAL EXAMPLE

Consider the following numerical example with the parameters fixed at the following values: $k = 20$, $a = 1$, and $b = 2$. The numerical results are summarized in Table 1. In the limiting case in which there are no road construction and

Table 1: Optimal Spacings and Net Benefits for Feeder Systems

$c = 0$

Number of feeders	d_i	Net benefits
1	(1.68)	223.33
2	(1.22, 4.81)	322.84
3	(0.965, 3.655, 6.78)	383.79

$c = 5$

Number of feeders	d_i	Net benefits
1	(1.85)	132.50
2	(1.48, 5.53)	155.34
3	(1.317, 4.737, 8.314)	148.03

maintenance costs, $c = 0$, the addition of feeders leads to continuously increasing net benefit, but at a constantly decreasing rate. As more and more feeders are added the spacing becomes increasingly dense and, in the limit, leads to zero spacing. Now consider the case in which $c = 5$. First, as one would expect, increasing construction and maintenance costs lead to shorter feeder roads located further from the port for any given number of feeders. When $c = 5$ the optimal number of feeder roads is $N = 2$. Both one and three road systems lead to an erosion of net benefits. Increasing N beyond three will necessarily reduce net benefits even further, eventually leading to negative net benefits.

SUMMARY AND CONCLUSION

In this paper it has been shown that it is possible to analytically determine the optimal structure of a feeder road system in a highly idealized case. The formal conditions for optimality of this system yield the common sense result that any optimally spaced feeder road system will result in the equation of the marginal costs of road construction and maintenance with the marginal incremental rent from road extension. However,the analytics are complicated by the various components to incremental rent due to the overlap of feeder road hinterlands. Moreover, it has been shown that the optimal structure of feeder road systems behaves as expected with variations in key parameters such as per unit road construction and maintenance cost. Future research will be directed towards a more thorough numerical analysis of the sensitivity of feeder road spacing to changes in the parameters a, b, c, and k. Of particular interest is the identification of conditions in which the optimal feeder road system makes a topological transition from N-1 to N feeder roads.

NOTES

1. This assumption has been made purely for geometric simplicity. Cherene, Neidercorn, and Song (1983) have analyzed an investment problem of this type using the Euclidean metric.
2. Only the area above the feeder road need be analyzed since the system is symmetric about PZ.
3. The case in which the new feeder is of inferior technology (i.e., in the range (a,b)) can be accommodated with only minor changes.
4. The case in which feeder roads are truncated before the limit of cultivation is more realistic but proves analytically to be much more difficult.

* I wish to acknowledge the assistance of Bruce Ralston and Brian McKee in the development of this paper. The graphics were prepared by Ole Heggen and Ken Josephson of the Department of Geography, University of Victoria.

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