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Facility Location Problems: Review, Description, and Analysis

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The purpose of this paper is (1) to provide a system by which location-decision problems can be categorized, and (2) to present the structure and the analysis of a representative subset of location problems that seem to be important. We have organized the review to convey to geographers and planners the normative approach of operations research toward the structuring and analysis of location problems. We have particularly focused on location problems under conditions of uncertainty. A representative sample of location problems, including both private- and public-sector problems, is discussed and categorized in the review.

The literature on location contains two related subjects: facility layout and facility location. The first deals with the determination of the configuration of certain types of facilities; the second concerns the location of the facilities. Throughout this paper we address facility location, a topic that has been the subject of analysis for centuries. Prior to the development of analytical approaches, solutions to facility-location problems were largely dependent upon subjective criteria as well as qualitative objectives and rules of thumb. Such approaches have gradually been replaced by a greater degree of reliance on quantitative analysis.

Two types of mathematical models were developed to assist facility-location decisions: descriptive and prescriptive (normative). In the former the model is used to describe the behavior of the system; in the latter the model is used to find solutions that, in some sense, are optimal. Typical normative models are mathematical programming models. In this paper we concentrate only on the analysis of normative models. These models were developed to analyze location decisions in the private and the public sector. We regard location problems in both sectors as falling into two groups, ordinary services and emergency services. In the public sector, ordinary services location problems arise, for example, when post offices, schools, highways, public buildings, parks, libraries, public housing, and environmental services are to be located. Public emergency services are provided by fire stations, police stations, emergency ambulances, hospitals, military bases, and so on. In the private sector, location decisions arise, for example, in manufacturing shops where the decision to be made is where to locate a new lathe. Other examples deal with the location of new warehouses relative to production facilities and customers, or even a component or components in an electrical network.

In general, the following elements are to be considered in classifying facility location problems: (1) solution-space characteristics, (2) distance measure, (3) new facility characteristics, (4) existing facility characteristics, (5) new and existing facility interactions, (6) objectives, (7) characteristics of the population served by the facilities, and (8) classes of problems.

Solution-Space Characteristics. An important factor in classifying a facility-location problem concerns the solution space. In some facility-location problems the solution space is one dimensional; such is the case where a facility is to be located on a road. More commonly, a two- or three-dimensional solution space exists. Additionally, the solution space may be constrained or unconstrained. An example of a constrained solution space is a network, that is, facilities may be located only on the network (both nodes and points on the arcs that join the nodes). Finally, one can consider either a discrete solution space, which typically consists of a finite number of possible locations, or a continuous solution space, where the space consists of infinite possible locations.

Distance Measure. The distance measure involved in the facility-location problems provides another basis for classifying such problems. One way to measure distance is according to a particular metric (norm). This is usually the way distance between two points on a plane is measured. One example is the Euclidean norm where $d_{ij}^2 = (X_i - X_j)^2 + (Y_i - Y_j)^2$, d_{ij} = the distance between points i and j, and X_i, Y_i = the coordinates in a rectangular system of the ith point. Another example is the metropolitan norm where $d_{ij} = |X_i - X_j| + |Y_i - Y_j|$. The discussion of such norms could easily be extended to cover other norms, such as the Tchebycheff L_∞ norm, so-called L_p norms, or combinations thereof (see Ward and Wendel, 1980). In location problems only one norm is traditionally considered, yet it is often difficult to determine which norm should be used to estimate accurate distances (Hansen, Penneur, and Thisse, 1980; Mehrez, forthcoming). Alternatively, distance measurement or time measurements can be defined along a network as d_{ij} = the length (time) of the shortest path from node i to node j.

New and Existing Facility Characteristics and the Interactions between Them. Facility characteristics and their mutual interactions were extensively discussed by Francis and White (1974). New and existing facilities can be subclassified as single or multiple facility problems. Furthermore, the new and existing facilities can be considered to occupy either point locations or area locations. If area location decisions are to be considered, the problem is often classified as a facility-layout problem. Additionally, we note that the number of new facilities can be either a decision variable or a given. Existing and new facilities can be either static or dynamic, as well as deterministic or probabilistic, depending on the nature of the problem. Finally, we recognize that the degree of interaction between existing and new facilities is an important characteristic of a facility location problem. Clearly, the degree of interaction varies from one problem to another, and often it is a function of the locations of the facilities.

Objectives. Another category commonly used to classify facility location problems concerns the objective function employed to evaluate alternative solutions. Both multiobjective and single-objective problems are considered in the literature. The formulation of objectives depends primarily on whether the location decisions are private or public, and whether an ordinary service or an emergency service is provided. A reasonable statement of the objective of a private decision maker is the maximization of profit. Public location decisions are often made in response to an objective that is unquantifiable in dollar terms. Different types of objectives that may arise in the public sector are listed by Ravelle, Marks, and Liebman (1970). One objective that can be used is the average distance or time traveled by those who utilize the facilities. Another possible objective may take into account the extent of service provided by the facilities, or the maximum or minimum distance or time between any facility and the population areas that it is intended to serve.

Characteristics of the Population to Be Served by the Facilities. The characteristics of the population that is to be served by the facilities is another factor that influences the location problem. This population is either static or dynamic in its size and its location. The need for service and the location of the population may not be known at the time the location decisions are undertaken. Finally, a priority structure may classify the given population.

Classes of Problems. The past two decades have witnessed an explosive growth in the literature on location problems. Location is considered to be one of the more profitable areas of applied operations research. Not only operations researchers but economists, urban planners, architects, regional scientists, and engineers revealed an interest in location problems. The purpose of this paper is not to cover the location area from a general point of view; at most we will present several location problems that appear in the operations research (O.R.) literature. Krarup and Pruzan (1983) recently claimed that the following four location-problem groups have played a particularly dominant role: p-center, p-median, simple plant, and quadratic assignment (also referred to as prototype). These four types of problems, along with other types as well, have been treated at length in the O.R. literature, both in textbooks and in hundreds of published papers. For example, the p-center and p-median are treated in Francis and White (1974), Christofides (1975), Jacobsen and Pruzan (1978), and Handler and Mirchandani (1979), and in the survey by Krarup and Pruzan (1979). Quadratic assignment problems are analyzed by Hillier (1963), Hillier and Connors (1966), Gilmore (1962), Lawler (1963), Francis and White (1974), and others. The simple plant location problem (SPLP) has been surveyed and discussed by Krarup and Pruzan (1983); these authors have shown the relationship between the SPLPs and other types of problems, including p-center, p-median, set-packing, set-covering, and set-partitioning problems as well as various possible extensions of the SPLPs. The SPLPs are a family of discrete, deterministic, single-criterion problems. Their possible extensions and variants may consider capacitated, dynamic, stochastic, continuous, multicriteria, multicommodity, piece-wise linear, and investment factors.

The purpose of this paper is to present for the reader unfamiliar with the location literature some basic features of the aforementioned problems. The focus in the presentation will be on the assumptions behind the structure of the problem, its applicability and limitations, its solution method, and finally the characteristics of the solution and the procedures by which it can be obtained. The problems selected below are a sample of location problems. The sample does not intend to cover all types of situations where location decisions are undertaken.

The Generalized Weber Problem

Historically, economic location analysis began with Alfred Weber (1929), who considered the location on a plane of a factory between two resources and a single market. Beginning with the formulations of Cooper (1967) and Kuhn and Kuenne (1962), interest in location analysis quickened. The last two works, which appeared independently, described an iterative process for solving the generalized Weber problem. The problem is to find the single point that minimizes the sum of the weighted Euclidean distances to that point. Formally, the objective is

$$\text{Min } Z = \sum_{i=1}^{n} W_i \left[(X_i - X_p)^2 + (Y_i - Y_p)^2\right]^{1/2} , \tag{1}$$

where

W_i = the weight attached to the ith point (goods demanded, population size, etc);

X_i, Y_i = the location of the ith point relative to some fixed cartesian coordinate system;

X_p, Y_p = the unknown coordinates of the central point p;

n = the number of points that are served; and

d_{ip} = the Euclidean distance from point i to central point p, where

$$d_{ip} = \left[(X_i - X_p)^2 + (Y_i - Y_p)^2\right]^{1/2} . \tag{2}$$

A very simple example of this problem is where Z is expressed as dollars per year, W_i is expressed as dollars per distance, and d_{ip} as the dimension distance per trip. Thus, if d_{ip} equals 15 miles per trip and W_i equals the product of $0.2 per mile and 350 trips per year, then $W_i d_{ip}$ equals $1,050 per year. If the cost per unit distance is constant, then the minimization problem often reduces to a determination of the location that minimizes distance. This problem describes a situation where the central facility might be a point in an electrical network to which a number of wires are to be connected; the location of the point that will minimize the total cost of wire is to be determined. This problem is also called the general Fermat problem or the Steiner-Weber problem, or the 1-median problem, and is basically a continuous, static, deterministic, one-facility, linear cost-minimization problem.

A similar problem is the rectilinear-distance, one-facility location problem. This problem arises, for example, in a situation where travel occurs along a set of aisles arranged in a rectangular pattern parallel to the walls of a building. This is the situation in most machine-location problems, and in such a situation the objective is the same as in equations where $d_{ip} = |X_i - X_p| + |Y_i - Y_p|$. The Euclidean problem can be solved by an iterative procedure suggested by Kuhn and Kuenne and by Cooper. The iterative procedure is guaranteed to converge onto the optimum location. An iterative procedure is required because there is no direct solution for X_p and Y_p in terms of the following equations:

$$\frac{\partial Z}{\partial X_p} = \Sigma \frac{W_i(X_i - X_p)}{d_{ip}} = 0 , \tag{3}$$

and

$$\frac{\partial Z}{\partial Y_p} = \Sigma \frac{W_i(Y_i - Y_p)}{d_{ip}} = 0 , \qquad (4)$$

where (3) and (4) are the partial differentiation with respect to X_p and Y_p. The procedure is to solve (3) and (4) for X_p and Y_p in terms of W_i, X_i, Y_i, and d_{ip}:

$$X_p = \sum_i \frac{W_i X_i}{d_{ip}} \Big/ \sum_i \frac{W_i}{d_{ip}}, \qquad (5)$$

$$Y_p = \sum_i \frac{W_i Y_i}{d_{ip}} \Big/ \sum_i \frac{W_i}{d_{ip}}, \qquad (6)$$

The value of d_{ip} is then recalculated via (2) and the procedure repeated until successive differences between values of X_p and between values of Y_p are negligible. A good starting point is the centroid or the center-of-gravity solution, namely, set the initials

$$X_p = \frac{\sum_i W_i X_i}{\Sigma W_i} \quad \text{and} \quad Y_p = \frac{\sum_i W_i Y_i}{\Sigma W_i} . \qquad (7)$$

The centroid is the solution of the so-called gravity problem that can be formulated as

$$\text{Min } Z = \sum_{i=1}^{n} W_i[(X_i - X_p)^2 + (Y_i - Y_p)^2] . \qquad (8)$$

In contrast to (1), the solution to the rectilinear-distance location problem can be found without using an iterative procedure. In fact, the optimum X_p coordinate (Y_p coordinate) is a point i such that no more than half-sum of the weights is to its left (below) and no more than half-sum of the weights is to its right (above). For illustrative purposes, consider the following problem:

Point	X-coordinate value	Y-coordinate value	Weight (W)
1	2	6	3
2	7	19	15
3	12	5	12
4	32	22	21

Clearly, for the above example the optimal solution is to locate X_p at 12 and Y_p at 19. The extensions of the 1-median problem are analyzed in the O.R. literature to cover situations where the problem is to locate multiple facilities; or the solution space is a network or any other discrete solution space; or a norm that is different from the Euclidean or the rectilinear is used; or finally, the problem is stated under uncertain or dynamic conditions. These extensions and variants of the problem that complicate the analysis will not be discussed here. (References that survey these extensions are listed above.)

A Simple Plant Location Problem

A problem that has attracted much attention in the O.R. location literature is the so-called simple plant location problem (SPLP); a comprehensive exposition of it has been provided by Krarup and Pruzan (1983). It is concerned with the location of plants or facilities (e.g. factories, warehouses, schools) so as to minimize the total cost of serving clients. The problem, which has a transparent structure, contributed to the formulation and the solution of complex planning problems. The basic problem, which is a uniobjective, discrete, static, deterministic, one-product, fixed-plus-linear costs-minimization problem, can be extended to include dynamic, multiobjective, stochastic, multiproduct and nonlinear cost characteristics. In addition, Krarup and Pruzan (1983) have shown that different problems, such as the p-center and p-median, are transformable to SPLP. Following the notations of Krarup and Pruzan, we provide here the formulation of the problem. The basic assumptions are (1) that there is a finite set of possible locations for establishing new facilities, and (2) that the facilities have unlimited capacity such that any facility can satisfy all demands. The problem is to minimize costs while satisfying all demands. The constituents of SPLP are:

m: the number of potential facilities indexed by $i, i \in I = (1,\ldots,m)$,
n: the number of clients indexed by j, $j \in J = (1,\ldots n)$,
f_i: the fixed cost of establishing facility i,
p_i: the per unit cost of operating facility i (including variable production and administrative costs etc.),
b_j: the number of units demanded by client j,
t_{ij}: the transportation cost of shipping one unit from facility i to client j.

It is customary to use the adjectives *open* and *closed* for designating the state of a facility. The cost of sending no units from a facility is zero (i.e. the facility is closed), while any positive shipment from the ith facility incurs a fixed cost f_i (the facility is open) plus costs $p_i + t_{ij}$ per unit produced at facility i and transported to the client j. We introduce the m + mn variables:

y_i: $y_i=1$ if facility i is open and 0 otherwise,
s_{ij}: number of units produced at facility i and shipped to client j.

The full-blooded SPLP is the mixed-integer program:

$$\min \sum_{i\in I} \sum_{j\in J} (p_i + t_{ij})s_{ij} + \sum_{i\in I} f_i y_i , \quad (9)$$

$$\sum_{i\in I} s_{ij} \geq b_j, \qquad j\in J, \quad (10)$$

$$k_i y_i - \sum_{j\in J} s_{ij} \geq 0, \qquad i\in I, \quad (11)$$

$$s_{ij} \geq o, \qquad i \quad I, j\in J, \quad (12)$$

$$y_i \; \{0,1\}, \qquad i\in I. \quad (13)$$

The m restrictions (11) are devices to ensure that the total fixed cost for a facility is incurred whenever positive shipments are made from it. The k's are positive constants, not less than the maximal outflow from the corresponding facilities. If all $p_i \geq 0$, $t_{ij} \geq 0$, no facility need ever ship more than the total amount demanded, and each k_i may be replaced by $\Sigma_{j\in J} b_j$. Similarly, because under the assumption $(p_i + t_{ij}) \geq 0$ it will never pay to ship a larger number of units to a client than demanded, the inequalities (10) can be replaced by equations.

Thus, according to Guignard and Spielberg (1977), "the SPLS is one of the simplest mixed integer problems which exhibits all the typical combinational difficulties of mixed (0−1) variables and at the same time has a structure that invites the application of various specialized techniques." In general the SPLP is termed an NP complete. It means that the amount of time required for solving the problem is not bounded by some polynomial function of the input length. Since the mid-sixties, a wide selection of different algorithms (including exact and heuristic approaches) have been tailored for SPLP with varying degrees of success, and numerous computer codes have been written, implemented, tested, and reported in the literature. In spite of the significant efforts that were invested to solve SPLP, earlier by heuristics and later by different exact methods such as cutting plans, dynamic programming, branch and bound, enumerative approaches, etc., the solutions to a significant proportion of real SPLP problems or location problems in general still cannot be successfully computed and solved.

Minimax and the Maximin Location Problems

Two types of problems extensively treated in the literature are the maximin and the minimax location problems. The maximin problem was first treated by Dasarthy and White (1980) for the Euclidean case, and subsequently by Drezner and Wesolowsky (1981). These authors studied the problem of locating a point that maximizes the minimum Euclidean distance from a given set of points. Three possible application areas for this problem were reported by Dasarthy and White:

1. The case of radar stations gathering intelligence on enemy ships over a specified region: Given the locations and the number of radar stations that monitor a specified region, the problem is to find the minimum (of the maximum) power required by the stations such that each point in the region is monitored by one or more of the stations.
2. Given m cities in a region S, the location of a highly polluting industry within S: One criterion in choosing the location could be that the amount of pollutant reaching any city is minimized.
3. Application in information theory: Find the point that maximizes its minimum distance from m known signals.

A related minimax problem is finding a point that minimizes the maximum Euclidean or rectilinear distance from a given set of points. This problem is known as "the minimum covering sphere problem," or the 1-center problem. In this section we will analyze the rectilinear problem under different structures of certainty and uncertainty. Actually, we will summarize the work of Carbone and Mehrez (1980) on the minimum maximum distance single-facility location problem applied to situations where the locations of prospective demand points are considered to be random variables. Two types of decisions are analyzed for this setting under the assumption of independent and identical normal distributions with the same means: locating on the basis of an expected value criterion, or adopting a wait-and-see policy. Through the concept of the expected value of perfect information (EVPI), it is shown for one-dimensional location decisions that a substantial reduction in maximum distance may be realized by the adoption of a wait-and-see policy.

The Basic Structure

Let (X_1,Y_1), (X_2,Y_2),...,(X_n,Y_n) denote n demand points on a plane. The problem of identifying the location (X,Y) of a single facility that minimizes the maximum rectilinear distance S from the facility to any of the n demand points may be written as Problem I:

$$\min_{x,y} (\max_{i}(|X_i - X| + |Y_i - Y|)). \qquad (14)$$

Given knowledge of the location of the demand points on the plane, the closed form solution of Problem I given by Elzinga and Hearn (1972) is

$$S^* = \max \left\{ \frac{\max (X_i+Y_i) - \min(X_i+Y_i)}{2}, \frac{\max(X_i-Y_i) - \min(X_i-Y_i)}{2} \right\}. \qquad (15)$$

In several instances the assumption of known fixed demand points does not hold. For example, consider the case of the location decision of a single fire station that is to serve n prospective residents of a new residential community. In such a setting, one can assume that at best the decision maker knows the expected future center of the community as well as the distance distribution of each demand point on the plane.

The purpose now is to evaluate two policies that may be followed in situations where, within the confines of Problem I, the locations of the n demand points are treated as random variables. More specifically, we examine in the same way as did Wesolowsky (1977) the expected loss in terms of maximum distance resulting from using an expected value criterion to locate the single facility as opposed to adopting a wait-and-see policy.

We assume that $X_1, X_2,...,X_n, Y_1, Y_2,...,Y_n$ are identical, pairwise independent, normally distributed random variables with mean 0 and variance σ^2. Without loss of generality, we set $\sigma^2 = 1$ to simplify the notation to come. We first examine the problem of locating on the basis of an expected value criterion. This may be formulated simply as Problem II:

$$\min E (\max (|X_i - X| + |Y_i - Y|)). \qquad (16)$$

The optimal solution of this problem is given by a result due to Carbone and Mehrez (1980) that states that the optimal location of the single facility under Problem II is at the (0,0) coordinate.

A second formulation, which is commonly referred to as the wait-and-see problem, implies that the decision maker first observes the actual location of the n demand points and then determines the optimal location of the facility. Clearly, the optimal value of a wait-and-see policy is the E(S*); that is, the expected value of the maximum of ranges of two independent samples each including n independent normal variates.

The Expected Value of Perfect Information

These two alternative policies may be adopted by a decision maker. Evaluation of them may be determined through the expected value of perfect information (EVPI), a concept introduced in the location literature by Wesolowsky (1977) for a one-dimensional "Weber" single-facility location problem.

EVPI is defined here as the benefit gained in terms of reducing the maximum distance through a wait-and-see policy. By Theorem 1 (Mehrez and Stulman, 1982) and the optimal value of a wait-and-see policy,

$$\text{EVPI} = E(\max_i (|X_i| + |Y_i|)) - E(S^*). \tag{17}$$

To provide insight into the value of EVPI without undue computational effort, we now assume for illustrative purposes that the demand points and the facility are to be located along a road. Given this assumption,

$$E(S^*) = \tfrac{1}{2}E(\max_i X_i - \min_i X_i) = E(\max_i X_i) \tag{18}$$

and

$$\text{EVPI} = E(\max_i |X_i|) - E(\max_i |X_i|). \tag{19}$$

Because it is known that the largest absolute deviation in a sample of size n taken from a symmetric distribution approaches the largest value in a sample of size 2n (see Gumbel, 1958, 95), $E(\max|X_i|)$ can be approximated by $E(\max_{1 \leq i \leq 2nxi})$. Hence,

$$\text{EVPI} \simeq E(\max_{1 \leq i \leq 2n} X_i) - E(\max_{1 \leq i \leq n} X_i). \tag{20}$$

We are given the following properties of the expected range as a differentiable function of a sample of size n taken from a symmetrical distribution: (1) the expected range increases in n, (2) this increase diminishes with increasing n. It is easily seen that (1) implies the nonnegativity of EVPI, and that according to (2), EVPI is decreasing in n. In light of the asymptotic properties of the normal distribution that are assumed here, one may then be led to believe that EVPI rapidly approaches zero. As a result, for a sufficient number of demand points, the best decision would be to locate the facility at the middle point of the road. This, however, is not the case. Looking at values tabulated by Harter (1961) for $n \leq 400$, which are reported in table 1, one sees that EVPI $\simeq$ 0.22 for sufficiently large n.

TABLE 7.1
Computation of $E(\max_i X_i)$ for Values of n

n	$E(\max_i X_i)$
25	1.96531
50	2.24907
100	2.50759
200	2.74604
400	2.96818

These results illustrate that, even under the stringent assumptions, a substantial improvement in the quality of service may be realized by a wait-and-see policy. One must take into account that no service would be initially provided to the first demand points through the adoption of such a policy.

The Maximal Covering Location Problem with Facility Placement on the Entire Plane

The purpose of this section is to analyze a continuous location problem that can be reduced to a discrete structure. A linear integer code can be used to solve the problem for a fairly large size area.

Recently Mehrez and Stulman (1982) discussed the maximal covering location problem with facility placement on the entire plane. The basic idea behind this problem is to fix the number and location of facilities to cover a given set of demand points, such that each demand point is served by a facility within a preset standard, $\bar{R}$, measured by a Euclidean norm. The problem was previously treated by Toregas (1971), White and Case (1974), and Schilling (1980) under an assumption that restricts the location of the facilities to a fixed finite set of possible locations. This assumption enables these authors to solve their problem efficiently by using a zero-one integer linear algorithm. Mehrez and Stulman (1982) have shown that an optimum solution for the entire plane problem can be found on a small finite set of points.

More specifically, suppose we are given a set of demand points, X_i, Y_i, $i=1,...,n$, and P is the largest number of facilities that can be located to serve these points. Mehrez and Stulman (1982, Theorem 1) have shown that under reasonable assumptions the optimal location of these facilities is at some intersection of two circles of radius $\bar{R}$ about two of the demand points. Under these assumptions we define by $j=1,...,J$ a possible point of a facility, where J is at most equal to $2\binom{n}{2}$. The Euclidean distance of point i from facility j is measured by R(i,j). Let a_{ij} be a scalar with value one if $R(i,j) \leq \bar{R}$ and zero otherwise. The maximal covering location problem on the entire plane is to place P facilities such that

$$\text{Max } Z = \sum_{i=1}^{n} W_i ,$$

subject to

$$\sum_{j=1}^{J} X_j \leq p, \tag{21}$$

whereby

$$W_i - \sum_{j=1}^{J} a_{ij}X_j \leq 0 \qquad (i=1,\ldots,n).$$

W_i and X_j are zero-one integer variables where X_j is given the value one if a facility is located at point j and zero otherwise. W_i is given the value one if there is at least one facility located at j, such that $R(i,j) \leq \bar{R}$ and zero otherwise. Actually

$$W_i = \text{Min}\left\{1, \sum_{j=1}^{J} a_{ij}X_j\right\}. \tag{22}$$

An Illustrative Example

Consider figure 1, which shows the location of n=20 demand points that are the centers of the small circles (indicated by 1−20). Around each point we have drawn a circle of radius $\bar{R}$ of 3.1 units and have marked all the intersection points (1-77). The maximal number of intersection points is $2\binom{20}{2} = 380$. However, in this case the actual number is smaller. The problem was solved by an integer linear program located on the MPOS package. Actually, we ran the problem for p=3 and 5, where p=5 was the minimal number of facilities that cover all the demand points. The solutions and the CPU time required to run the problems are presented in table 2.

Our experience with this formulation (for n about 20) has shown up to a 100-fold increase in computer efficiency compared to doing complete enumeration. The exact increase in efficiency is difficult to express because it is dependent on the particular example.

FIGURE 7.1
The Location of the Demand Points

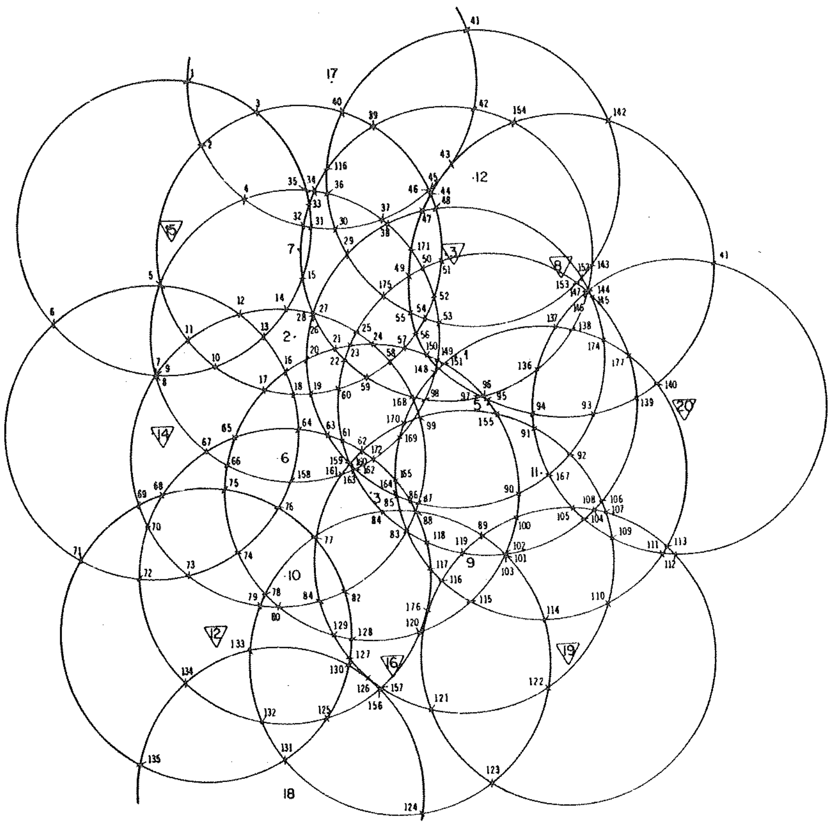

TABLE 7.2
The Results of an Illustrative Example

p	Z	Facilities	Demand Points That Are Served	CPU TIME	Optimal Interaction
3	16	55,76,119	1,2,3,4,5,6,7,8,9,10,11,12, 13,14,16,19	54.9	65
5	20	1,18,55, 108,133	all	69.4	77

This problem has a formulation similar to that of the discrete formulation of the maximum covering problem and the minimum facility location problem discussed by Toregas (1971), White and Case (1974), and Schilling (1980). We note that equation (21) can be easily modified to consider situations where weights that measure the relative importance of each demand point are added to the objective function or constraints on the capacity of each facility are imposed.

A Quadratic Assignment Location Problem

Francis and White (1974) considered the problem of assigning facilities to sites when there is an interchange between points of new facilities. Furthermore, exactly one facility is to be assigned to each site. The sites might be rooms in a plant, and the facilities might be departments to be assigned to the rooms. The problem of assigning new facilities to sites when there is an interchange between new facilities is referred to as a quadratic assignment location problem.

This location problem was formulated by Francis and White as follows. Let C_{ikjh} denote the annual cost of having facility i located at site k and facility j located at site h. Also, let the decision variable X_{ik} equal one if facility i is located at site k and equal zero, otherwise. If there are n new facilities and sites, we wish to minimize

$$f(x) = \tfrac{1}{2} \sum_{i=1}^{n} \sum_{k=1}^{n} \sum_{j=1}^{n} \sum_{h=1}^{n} C_{ikjh} X_{ik} X_{jh} ,$$

subject to

$$\sum_{i=1}^{n} X_{ik} = 1, \quad k=1,\dots,n \tag{23}$$

$$\sum_{k=1}^{n} X_{ik} = 1, \quad i=1,\dots,n$$

$$X_{ik} = 0 \text{ or } 1, \text{ for all } i,k.$$

Notice that if facility i is located at site k and facility j is located at site h then X_{ik} and X_{jh} both equal 1 and the cost term C_{ikjh} is included in the total cost calculation. The first set of constraints ensures that exactly one facility is assigned to each site; the second set of constraints ensures that each facility is assigned to exactly one site. We note that this problem can be solved by either heuristic procedures, Hillier (1963), or exact solutions based on the work of Gilmore (1962) and of Lawler (1963).

A basic problem in the application of location models stems from the gap between the formal structure of the model and the characteristics of typical location problems that are dealt with in reality. This problem is especially acute in dealing with public-sector problems. The main goal of this paper is to aid planners and geographers in evaluating and selecting models appropriate to the reality with which they deal.

In conclusion we would like to emphasize that this paper represents neither an independent research effort nor an all-inclusive summary of location problems. It is an attempt to highlight various attempts to use theoretical models in analyzing and solving location problems in reality.

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